A modal logic for belief functions on MV-algebras

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Algebraic Semantics for Uncertainty and Vagueness
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3 Semantics for the modal logics $FP(\Lambda_k, \mathcal{L})$ and $FP(C\Lambda_k, \mathcal{L})$
   - Probabilistic models
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Belief functions on Boolean algebras

Let $X$ be a finite set (the *frame of discernment*) and let $m : \mathcal{P}(X) \rightarrow [0, 1]$ be a map such that

$$\sum_{A \subseteq X} m(A) = 1, \text{ and } m(\emptyset) = 0.$$ 

The map $m$ is called the *mass assignment*, and the *belief function* over $\mathcal{P}(X)$ defined from $m$ is the map $b_m : \mathcal{P}(X) \rightarrow [0, 1]$ such that for every $A \in \mathcal{P}(X)$

$$b_m(A) = \sum_{B \subseteq A} m(B).$$
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Belief functions

A subset $A \subseteq X$ such that $m(A) > 0$ is said to be a *focal element*, and clearly the belief function $b_m$ is defined from the restriction of $m$ over the focal elements.

Notice that every mass assignment $m$ on $\mathcal{P}(X)$ induces a probability measure $\mathbb{P}_m$ on $\mathcal{P}(\mathcal{P}(X))$. Therefore, given a mass assignment $m$, for every $A \subseteq X$, we can equivalently define

$$b_m(A) = \mathbb{P}_m(\beta_A),$$

where $\beta_A = \{B \mid B \subseteq A\}$, or as membership function on $\mathcal{P}(\mathcal{P}(X))$

$$\beta_A : B \in \mathcal{P}(X) \mapsto \begin{cases} 1 & \text{if } B \subseteq A \\ 0 & \text{otherwise}, \end{cases}$$
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Belief functions on MV-algebras (1)

In order to generalize belief functions to MV-algebras of functions, Kroupa provides the following approach: Consider a finite set $X$ and let $M$ be the MV-algebra of functions $[0, 1]^X$ (i.e. fuzzy subsets of $X$). For every $a \in M$, let $\hat{\rho}_a : \mathcal{P}(X) \to [0, 1]$ be defined as follows: for every $B \subseteq X$,

$$\hat{\rho}_a(B) = \min\{a(x) : x \in B\}.$$

The map $\hat{\rho}_a$ generalizes $\beta_A$ because if $a$ is a Boolean function, then $\hat{\rho}_a = \beta_a$.

**Definition**

A Kroupa belief function is a map $\hat{b} : [0, 1]^X \to [0, 1]$ such that, for every $a \in [0, 1]^X$,

$$\hat{b}(a) = \hat{s}(\hat{\rho}_a),$$

where $\hat{s} : [0, 1]^{\mathcal{P}(X)} \to [0, 1]$ is a state.
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  \[ \hat{b}(a) = \sum_{B \subseteq X} \hat{\rho}_a(B) \cdot \hat{s}(B). \]

- The restriction of $\hat{s}$ to $\mathcal{P}(X)$ (call it $\hat{m}$) is a classical mass assignment. Therefore a focal element is any $B \subseteq X$ such that $\hat{m}(B) > 0$. That is, focal elements are classical sets.
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We generalize Kroupa’s belief functions on $[0, 1]^X$ by allowing focal elements to be elements of the same MV-algebra $[0, 1]^X$. What we need to generalize is the map $\rho$ that measures the degree of inclusion between fuzzy sets.

For every $a \in [0, 1]^X$ we define $\rho_a : [0, 1]^X \to [0, 1]$ as follows: for every $b \in [0, 1]^X$,

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For every $a \in [0, 1]^X$, the map $\rho_a$ generalizes $\hat{\rho}_a$ because for every crisp subset $B$ of $X$, $\rho_a(B) = \hat{\rho}_a(B)$.

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## Kroupa approach vs. Our approach

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Logical approach: $FP(\Lambda_k,Ł)$ and $FP(C\Lambda_k,Ł)$

The above definitions suggest that a logic for belief functions on MV-events can be introduced by expanding the language of Łukasiewicz logic by two unary modalities:

- A modality $\Box$ whose interpretation is intended to capture the behavior of the measure of inclusion $\hat{\rho}$, or $\rho$, we are dealing with.
- A modality $Pr$ that respects the axioms of states on MV-algebras.

Finally we interpret the belief of $\varphi$ by $Pr(\Box \varphi)$ (a similar approach was used by Godo, Hájek and Esteva to deal with belief functions over Boolean events).
Consider the $k$-valued Łukasiewicz logic expanded with rational truth constants $Ł^c_k$.

A $Ł^c_k$-Kripke model is a triple $⟨W, e, R⟩$ where:

- $W$ is a non-empty set of possible worlds,
- for every possible world $w$, $e(·, w)$ is a truth-evaluation of $Ł^c_k$ into $S_k$,
- $R : W × W → S_k$ is an accessibility relation.

We denote by $Fr$ the class of $Ł^c_k$-Kripke models.

If the accessibility relation $R$ is crisp (i.e. $R : W × W → \{0, 1\}$), then the model is called a classical Kripke model, and we will denote by $CFr$ the class of all classical Kripke models.
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The modal logics $\Lambda_k$ and $C\Lambda_k$

Bou, Esteva, Godo and Rodriguez introduce the logics $\Lambda(Fr,\mathcal{L}_k^c)$ and $\Lambda(CFr,\mathcal{L}_k^c)$ by enlarging the language of $\mathcal{L}_k^c$ by a unary modality $\Box$, and defining well formed formulas as usual. Now we are going to consider the two fragments $\Lambda_k$ and $C\Lambda_k$ of $\Lambda(Fr,\mathcal{L}_k^c)$ and $\Lambda(CFr,\mathcal{L}_k^c)$, whose well formed formulas have unnested occurrences of $\Box$, so to keep the modal logic to be locally finite.

Given a formula $\phi$, and a (classical, $\mathcal{L}_k^c$)-Kripke model $K = \langle W, e, R \rangle$, for every $w \in W$, we define the truth value of $\phi$ in $K$ at $w$ as follows:

- If $\phi$ is a formula of $\mathcal{L}_k^c$, then $\|\phi\|_w = e(\phi, w)$,
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- If \( \phi \) is a compound formula, its truth value is computed the truth functionality.
Then the logic $\Lambda_k$ has the following axioms:

1. all the axioms for $\mathcal{L}_k^c$;
2. $\square 1$;
3. $(\square \varphi \land \square \psi) \rightarrow \square (\varphi \land \psi)$;
4. $\square (\overline{r} \rightarrow \varphi) \leftrightarrow (\overline{r} \rightarrow \square \varphi)$.

The rules of $\Lambda_k$ are Modus Ponens, $\{\varphi, \varphi \rightarrow \psi\} \vdash \psi$; and Monotonicity, $\varphi \rightarrow \psi \vdash \square \varphi \rightarrow \square \psi$.

The logic $C\Lambda_k$ is $\Lambda_k$ plus the axiom $\{\square (\varphi \rightarrow \psi) \rightarrow (\square \varphi \rightarrow \square \psi)\}$.

- The logic $\Lambda_k$ is sound and complete w.r.t. $Fr$.
- The logic $C\Lambda_k$ is sound and complete w.r.t. $CFr$. 

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Probabilistic logics over $\Lambda_k$, and $C\Lambda_k$

The logics $FP(\Lambda_k, \mathcal{L})$ and $FP(C\Lambda_k, \mathcal{L})$ have a language obtained by expanding the language of $\Lambda_k$ by a unary modality $\sigma$. Formulas are those of $\Lambda_k$, plus the class $\mathcal{F}^\sigma$ that includes $\mathcal{F}^\Box$ and satisfies the following: for every $\psi \in \mathcal{F}^\Box$, $\sigma\psi \in \mathcal{F}^\sigma$, and $\mathcal{F}^\sigma$ is closed under the connectives of $\mathcal{L}$.

Axioms and rules of $FP(\Lambda_k, \mathcal{L})$ are as follows:

1. All the axioms and rules of $\Lambda_k$ restricted to the formulas in $\mathcal{F}^\Box$;
2. The following axioms for $\sigma$ (cf. [FG07]):
   1. $\sigma \top$.
   2. $\sigma(\neg \varphi) \leftrightarrow \neg \sigma(\varphi)$.
   3. $\sigma(\varphi \oplus \psi) \leftrightarrow [(\sigma(\varphi) \rightarrow \sigma(\psi \& \varphi)) \rightarrow \sigma(\psi)]$.
3. The rule of Necessitation, $\varphi \vdash \sigma(\varphi)$.

Axioms and rules of $FP(C\Lambda_k, \mathcal{L})$ are as above, replacing the axioms of $\Lambda_k$ for the formulas in $\mathcal{F}^\Box$, with those of $C\Lambda_k$. 
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Outline

1. Belief functions

2. Logical approach: $FP(\Lambda_k, \mathcal{L})$ and $FP(C\Lambda_k, \mathcal{L})$

3. Semantics for the modal logics $FP(\Lambda_k, \mathcal{L})$ and $FP(C\Lambda_k, \mathcal{L})$
   - Probabilistic models
   - Belief function models
Probabilistic models

The first kind of models for $FP(\Lambda_k, \mathcal{L})$ and $FP(C\Lambda_k, \mathcal{L})$ are defined as follows:

Definition

A probabilistic Kripke model is a system

$$M = \langle W, e, R, s \rangle$$

such that its reduct $\langle W, e, R \rangle$ is a $\mathcal{L}^c_k$-Kripke model, and $s : \mathcal{F}^\Box_M \to [0, 1]$ is a state, where $\mathcal{F}^\Box_M = \{\|\varphi\|_M : w \in W \mapsto \|\varphi\|_{M, w} : \varphi \in \mathcal{F}^\Box\}$.

A probabilistic $\mathcal{L}^c_k$-Kripke model such that its reduct $\langle W, e, R \rangle$ is a classical Kripke model, is called a probabilistic classical Kripke frame.
Let $M = \langle W, e, R, s \rangle$ be a probabilistic $\mathcal{L}_k^c$ (classical) Kripke model. For every $\Phi \in \mathfrak{F}^\sigma$, and for every $w \in W$, we define the truth value of $\Phi$ in $M$ at $w$ inductively as follows:

- If $\Phi \in \mathfrak{F}^\Box$, then its truth value $\|\Phi\|_{M,w}$ is evaluated in the fragment $\langle W, e, R \rangle$ as we defined in the previous section.
- If $\Phi = \sigma \psi$, then $\|\sigma \psi\|_{M,w} = s(\|\psi\|_M)$.
- If $\Phi$ is a compound formula, its truth values is computed by truth functionality.

**Theorem (Probabilistic completeness)**

1. The logic $FP(\Lambda_k, \mathcal{L})$ is sound and finitely strong complete with respect to the class of probabilistic $\mathcal{L}_k^c$-Kripke models.
2. The logic $FP(\mathcal{C} \Lambda_k, \mathcal{L})$ is sound and finitely strong complete with respect to the class of probabilistic classical Kripke models.
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Belief function models

Definition

The set of belief formulas (or B-formulas) is the subset of $\mathcal{F}^\sigma$ defined as follows: atomic belief formulas are those in the form $\sigma \Box \psi$ (where of course $\psi$ is a formula in $\mathcal{L}^c_k$), that will be henceforth denoted by $B(\psi)$; compound belief formulas are defined from atomic ones using the connectives of $\mathcal{L}$. The set of belief formulas will be denoted by $\mathcal{F}^B$.

Let now $\Omega$ be the set of all the evaluations of $\mathcal{L}^c_k$, over the (finite) set of propositional variables $V$, i.e. $\Omega = (S_k)^V$. For every formula $\varphi$ without occurrences of modalities (i.e. $\varphi$ is a formula in the language of $\mathcal{L}^c_k$), let $\|\varphi\|_\Omega : \Omega \rightarrow S_k$ be defined as $\|\varphi\|_\Omega(w) = w(\varphi)$. 
Definition

A (Kroupa) belief function model is a pair $N = (\Omega, m)$ where $\Omega$ is as above, and $m : (S_k)^{\Omega} \rightarrow [0, 1]$ ($m : \{\{0, 1\}\}^{\Omega} \rightarrow [0, 1]$) satisfies $\sum_{f \in (S_k)^{\Omega}} m(f) = 1$, and $m(\emptyset) = 0$. Then the corresponding belief function $bel_m$ is defined as usual: for every formula $\varphi$,

$$bel_m(\varphi) = \sum_{g \in (S_k)^{\Omega}} \rho_{\|\varphi\|_{\Omega}}(g) \cdot m(g).$$

For every belief formula $\Phi$, and every belief function model $N = (\Omega, m)$, $\Phi$ is evaluated into $N$ as follows:

- If $\Phi = B(\varphi)$ is atomic, then $\|B(\varphi)\|_N = bel_m(\varphi)$.
- If $\Phi$ is compound, then $\|\Phi\|_N$ is computed by truth functionality as usual.
Let $\Phi$ be a belief formula. Then for every (Kroupa) belief function model $D = (\Omega, m)$ there exists a (classical) probabilistic Kripke model $K = (W, e, R, s)$ such that $\Phi, \|\Phi\|_M = \|\Phi\|_D$, and vice-versa.

Hence, if we limit to belief formulas, and belief theories, then $FP(\Lambda_k, \mathcal{L})$ is sound and finitely complete with respect to the class of belief models. An analogous result holds for $FP(C\Lambda_k, \mathcal{L})$ with respect to Kroupa belief models.
**Theorem**

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