Continuous approximations of MV-algebras with product and product residuation

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Abstract. Recently, MV-algebras with product have been investigated from different points of view. In particular, in [EGM01], a variety resulting from the combination of MV-algebras and product algebras (see [H98]) has been introduced. The elements of this variety are called LΠ-algebras. Even though the language of LΠ-algebras is strong enough to describe the main properties of product and of Lukasiewicz connectives on [0, 1], the discontinuity of product implication introduces some problems in the applications, because a small error in the data may cause a relevant error in the output. In this paper we try to overcome this difficulty, substituting the product implication by a continuous approximation of it. The resulting algebras, the LΠq-algebras, are investigated in the present paper. In this paper we give a complete axiomatization of the quasivariety obtained in this way, and we show that such quasivariety is generated by the class of all LΠq-algebras whose lattice reduct is the unit interval [0, 1] with the usual order.

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1 Introduction

MV-algebras with product have been widely investigated by many authors, [DND01], [Mo00], [EGM01], [M01] and [MP02] and with many motivations, arising from algebra, algebraic geometry and from the theory of many-valued control. While an axiomatization of the variety of MV-algebras with product generated by [0, 1] with the Lukasiewicz operators and with product seems to be very problematic, the presence of product residuation simplifies the situation. Consider the structure [0, 1]LΠ = ([0, 1], ⊕, ¬, ·, →π, 0, 1), where · denotes ordinary product, and the remaining operations are defined as follows:

\[ x \oplus y = \min\{x + y, 1\}, \quad ¬x = 1 - x, \quad x \rightarrow^{\pi} y = \begin{cases} y & \text{if } x > y \\ 1 & \text{otherwise} \end{cases} \]

Then [0, 1]LΠ generates a variety which can be axiomatized by a finite number of equations. The members of such variety, called LΠ-algebras have been deeply investigated, [Mo00], [EGM01], [M01] and [MP01].
A negative counterpart for the expressiveness of the language of LΠ-algebras is the loss of continuity of the truth functions of formulas, due to the fact that the truth function of \( \to \) is not continuous in \((0, 0)\); this may cause problems when using LΠ in the treatment of approximate data, because a small error in the data may cause relevant errors in the output.

These observations constitute the main motivation for an investigation of a class of algebras in which the discontinuous product implication is replaced by a continuous approximation of it. The idea is the following: we fix a positive number \( q \) (the intuition is that \( q \) is greater than 0 but very close to 0), and we replace product implication \( \to \) by the operation \( \to_q \) defined by \( x \to_q y = (x \lor q) \to y \). Note that \( \to_q \) defined in this way is continuous.

The present paper is devoted to an investigation of the general properties of LΠ\(_q\)-algebras. These algebras are introduced in Section 3. In this section we prove some general properties of these structures. For example, LΠ\(_q\)-algebras constitute a quasivariety, but not a variety. Moreover, every LΠ\(_q\)-algebra is isomorphic to a subdirect product of a family of linearly ordered LΠ\(_q\)-algebras, and is a subalgebra of a LΠ\(_q\) algebra obtained from a LΠ-algebra letting \( x \to_q y = (x \lor q) \to y \), where \( q \) denotes a suitably chosen constant. Finally, in Section 4 we prove that the class of LΠ\(_q\)-algebras is generated as a quasivariety by the class of all LΠ\(_q\)-algebras whose lattice reduct is \([0, 1]\) with the usual order.

2 Preliminaries

Definition 2.1 (see e.g. [BF00]). A hoop is an algebra \( \langle H, \circ, \to, 1 \rangle \) such that \( \langle H, \circ, 1 \rangle \) is a commutative monoid, and \( \to \) is a binary operation such that the following identities hold:

\[
x \to x = 1, \quad x \to (y \to z) = (x \circ y) \to z \quad \text{and} \quad x \circ (x \to y) = y \circ (y \to x).
\]

A Wajsberg hoop is a hoop satisfying the identity \( (x \to y) \to y = (y \to x) \to x \).

A bounded hoop is a hoop equipped with a constant 0 such that \( 0 \to x = 1 \).

A Wajsberg algebra is a bounded Wajsberg hoop.

The monoid operation of a Wajsberg algebra is usually denoted by \( \circ \). In the sequel, given a Wajsberg algebra, we write \( \neg x \) for \( x \to 0 \), \( x \oplus y \) for \( x \circ y \to (x \circ y) \to y \), \( x \land y \) for \( x \circ (x \to y) \), \( x \lor y \) for \( (x \to y) \to y \), and \( x \leq y \) for \( x \to y = 1 \). Note that \( \leq \) is a distributive lattice order, and \( \lor \) and \( \land \) are the corresponding operations of join and meet. We inductively define \((n)x\) and \(x^{(n)}\) by:

\[
(0)x = 0, \quad (n+1)x = (n)x \oplus x, \quad x^{(0)} = 1, \quad x^{(n+1)} = x^{(n)} \circ x.
\]

Wajsberg algebras constitute a variety generated by \([0, 1]_W = ([0, 1], \circ, \to, 0, 1)\).

If \( \langle A, \circ, \to, 0, 1 \rangle \) is a Wajsberg algebra, then the structure \( \langle A, \circ, \neg, 0, 1 \rangle \) is called a MV-algebra. Every Wajsberg algebra is termwise equivalent to a MV-algebra ([H98]). Thus we will often identify a Wajsberg algebra and the corresponding MV-algebra.
In attempting to axiomatize the class of LΠ-algebras, in [Mo00] the concept of PMV-algebra has been introduced.

**Definition 2.2** A PMV-algebra is an algebra \( A = \langle A, \oplus, \neg, \cdot, 0, 1 \rangle \) such that:

\( \langle A, \oplus, \neg, 0, 1 \rangle \) is a MV-algebra.

\( \langle A, \cdot, 1 \rangle \) is a commutative monoid.

For all \( x, y, z \in A \) one has: \( x \cdot (y \oplus z) = (x \cdot y) \oplus (x \cdot z) \), where \( x \oplus y = \neg \neg x \oplus y \).

A LΠ-algebra is an algebra \( A = \langle A, \oplus, \neg, \cdot, \to_\pi, 0, 1 \rangle \) such that \( \langle A, \oplus, \neg, 0, 1 \rangle \) is a PMV-algebra, \( \langle A, \cdot, \to_\pi, 0, 1 \rangle \) is a bounded hoop, and letting \( \neg_\pi x = x \to_\pi 0 \) and \( \Delta(x) = \neg_\pi \neg x \), the following equations hold:

\[ x \to_\pi y \leq x \to y. \]
\[ x \land \neg_\pi x = 0 \]
\[ \Delta(x) \oplus \Delta(x \to y) \leq \Delta(y) \]
\[ \Delta(x) \leq x \]
\[ \Delta(\Delta(x)) = \Delta(x) \]
\[ \Delta(x \lor y) = \Delta(x) \lor \Delta(y) \]
\[ \Delta(x) \lor \neg \Delta(x) = 1 \]
\[ \Delta(x \to y) \leq x \to_\pi y. \]

A LΠ\(_{1\frac{1}{2}}\)-algebra is a LΠ-algebra with an additional constant \( \frac{1}{2} \) satisfying \( \frac{1}{2} = \neg_\pi \frac{1}{2} \).

**Notation.** In the sequel we will omit the symbol \( \cdot \) when there is no danger of confusion. Moreover we inductively define \( x^n \) by \( x^0 = 1, \; x^{n+1} = x^n \cdot x \).

In [Mo00], Lemma 2.11 and Theorem 5.1, the following is shown:

**Proposition 2.3**

(i) Every PMV-algebra is isomorphic to a subdirect product of a family of linearly ordered PMV-algebras.

(ii) A PMV-algebra and its underlying Wajsberg algebra have the same congruences.

**Definition 2.4** (Cf [BKW77]). A lattice-ordered ring is a structure

\[ R = \langle R, +, -, \times, \lor, \land, 0 \rangle \]

such that:

(i) \( R = \langle R, +, -, \times, 0 \rangle \) is a ring.
(ii) $\mathcal{R} = (R, \vee, \wedge)$ is a lattice.

(iii) Let $\leq$ denote the partial order induced by $\vee$ and $\wedge$. Then $x \leq y$ implies $x + z \leq y + z$, and $x, y \geq 0$ implies $x \times y \geq 0$.

An $f$-ring is a lattice-ordered ring which is isomorphic to a subdirect product of linearly ordered lattice-ordered rings.

A strong unit of a lattice-ordered ring $R$ is an element $u \in R$ such that $u \times u \leq u$, and for all $a \in R$ there is $n \in \mathbb{N}$ such that $a \leq nu$, where $nu = u + \ldots + u$.

A commutative unitary $f$-ring with strong unit (for short: a c-s-u-f-ring) is a commutative $f$-ring with a unit for product which is also a strong unit.

In [DND01] the authors define a functor $\Gamma_R$ from the category of lattice-ordered rings with strong unit into a category of algebras, called product MV-algebras. Here we describe the restriction of $\Gamma_R$ to c-s-u-f-rings, which turns-out to be a functor from the category of c-s-u-f-rings into the category of PMV-algebras.

**Definition 2.5** The functor $\Gamma_R$ is defined as follows:

(i) Let $\mathcal{R} = (R, +, -, \times, \vee, \wedge, 0)$ be a c-s-u-f-ring, and let $u$ be the unit of $\mathcal{R}$ (which by definition is also a strong unit). Then $\Gamma_R(\mathcal{R})$ denotes the structure $\langle [0, u] \otimes \neg, \cdot \rangle$, where $[0, u] = \{x \in R : 0 \leq x \leq u\}$, $x \otimes y = (x + y) \wedge 1$, $\neg x = u - x$, and $\cdot$ is the restriction of $\times$ to $[0, u]$.

(ii) Let $\mathcal{R}, \mathcal{R}'$ be lattice-ordered rings, and let $h$ be a morphism (i.e., a homomorphism) from $\mathcal{R}$ into $\mathcal{R}'$. Then $\Gamma_R(h)$ is defined as the restriction of $h$ to $\Gamma_R(\mathcal{R})$. (Note that $\Gamma_R(h)$ is a homomorphism from $\Gamma_R(\mathcal{R})$ into $\Gamma_R(\mathcal{R}')$).

In [Mo02], as a special case of a result contained in [DND01], Theorem 4.2, the following is shown:

**Proposition 2.6** $\Gamma_R$ is an equivalence between the category of c-s-u-f-rings and the category of PMV-algebras.

## 3 LI$\Pi_q$ algebras

**Definition 3.1** A LI$\Pi_q$-algebra is a structure $\mathcal{A} = \langle A, \oplus, \neg, \cdot, \rightarrow_q, q, 0, 1 \rangle$ where $\langle A, \oplus, \neg, \cdot, 0, 1 \rangle$ is a PMV-algebra, $q$ is a constant, and $\rightarrow_q$ is a binary operation such that the following conditions hold:

(A1) $q \leq \neg q$

(A2) $x \rightarrow_q y = (x \vee q) \neg q y$

(A3) $(x \vee q)(x \rightarrow_q y) = (x \vee q) \wedge y$

(A4) $q \rightarrow_q (qx) = x$
If \( x^2 = 0 \) then \( x = 0 \)

**Examples.** Let \( A = \langle A, \oplus, \neg, \cdot, \rightarrow_q, 0, 1 \rangle \) be a linearly ordered \( L\Pi \)-algebra with more than two elements, and let \( q \in A \setminus \{0\} \) with \( q \leq \neg q \). Define \( x \rightarrow_q y = (x \lor q) \rightarrow q y \).

Thus we may assume without loss of generality \( \beta < \alpha \).

Now if \( q = 1/2 \). Then it is easily seen that \( A \) contains \( 0 \) and \( 1 \), and is closed under \( \ominus, \neg, \cdot \) and \( \rightarrow_q \) defined by \( x \rightarrow_q y = (x \lor \frac{1}{2}) \rightarrow \neg y \).

We verify e.g. closure under \( \cdot \) and under \( \rightarrow \).

Let \( \alpha, \beta \in A \), where \( \alpha = \frac{p+\varepsilon^2 p(\varepsilon)}{q+\varepsilon^2 q(\varepsilon)} \), and \( \beta = \frac{r+\varepsilon^2 r(\varepsilon)}{s+\varepsilon^2 s(\varepsilon)} \).

Then \( \alpha \beta = \frac{p+\varepsilon^2 P(\varepsilon)}{q+\varepsilon^2 R(\varepsilon)} \).

Hence \( A \) is closed under \( \cdot \).

Now if \( \alpha \neq 1/2 \), then \( \alpha \rightarrow_q \beta = 1 \in A \).

Thus we may assume without loss of generality \( \beta < \alpha \) and \( \frac{1}{2} < \alpha \).

In this case, \( \alpha \rightarrow_q \beta = \frac{\beta}{\alpha} = \frac{r+\varepsilon^2 T(\varepsilon)}{q+\varepsilon^2 U(\varepsilon)} \).

However, \( A \) is not a \( \rightarrow_q \)-reduct of a \( L\Pi \)-algebra, because \( A \) is not closed under \( \rightarrow \).

To see this, note that both \( \varepsilon^2 \) and \( \varepsilon^3 \) have the form \( \frac{p+\varepsilon^2 p(\varepsilon)}{q+\varepsilon^2 q(\varepsilon)} \), take \( p = 0 \), \( q = 1 \), and \( Q(x) = 0 \); then \( \varepsilon^2 \) is obtained letting \( P(x) = 1 \), and \( \varepsilon^2 \) is obtained letting \( P(x) = x \). Hence \( \varepsilon^2 \) and \( \varepsilon^3 \) are elements of \( A \). However, \( \varepsilon^2 \rightarrow \neg \varepsilon^3 = \varepsilon \notin A \).

Indeed, \( \varepsilon = \frac{p+\varepsilon^2 p(\varepsilon)}{q+\varepsilon^2 q(\varepsilon)} \) would imply \( p+\varepsilon^2 P(\varepsilon) = q\varepsilon + \varepsilon^2 Q(\varepsilon) \), and finally \( p = q = 0 \). But \( q = 0 \) is excluded by the definition of \( A \).

From Definition 3.1 it follows that the \( L\Pi_q \)-algebras form a quasivariety. However:

**Theorem 3.2** The class of \( L\Pi_q \)-algebras does not constitute a variety.

**Proof.** Let \( [0, 1]^* \), \( q, \varepsilon \) and \( \rightarrow_q \) be as in the example above, and let us consider the structure \( A = \langle [0, 1]^*, \oplus, \neg, \cdot, \rightarrow_q, 0, 1, q \rangle \) (with \( \oplus, \neg, \cdot \) defined in the obvious way).

It is readily seen that \( A \) is a \( L\Pi_q \)-algebra.

Define for \( x, y \in A \), \( x \theta y \) if there is \( k \in \mathbb{N} \) such that \( |x - y| \leq (k)\varepsilon^2 \), where \( |x - y| = (x \ominus y) \lor (y \ominus x) \). It is easy to see that \( \theta \) is a congruence of PMV-algebras. We show that \( \theta \) is compatible with \( \rightarrow_q \). It is sufficient to prove that if \( |x - y| \leq (k)\varepsilon^2 \) then:

(a) \( |(x \rightarrow_q z) - (y \rightarrow_q z)| \leq (4k)\varepsilon^2 \) and (b) \( |(z \rightarrow_q x) - (z \rightarrow_q y)| \leq (4k)\varepsilon^2 \).

The proof of (a) splits into the following cases:
If \( x \lor q \leq z \) and \( y \lor q \leq z \), the claim is trivial.

If \( x \lor q \leq z \) and \( y \lor q > z \), then 
\[
\begin{align*}
(\forall v \lor q)z & \leq (\forall v \lor q)(x \lor q) - (y \lor q, z) \\
& \leq \left( \frac{x-v}{y-v} \right) q \leq \left( 2k \right) \varepsilon^2 \leq (4k) \varepsilon^2.
\end{align*}
\]

It follows that the class of \( L_{\Pi} \) one has
\[
\{ q \mid q \leq z \} \leq \left( \frac{x-q}{y-q} \right) q \leq (4k) \varepsilon^2.
\]

The proof of (b) splits into the following cases:

- If \( z \lor q \leq x \) and \( z \lor q \leq y \), the claim is trivial.
- If \( z \lor q \leq x \) and \( z \lor q > y \), then 
\[
\begin{align*}
(\forall v \lor q)z & \leq (\forall v \lor q)(x \lor q) - (z \lor q, y) \\
& \leq \left( \frac{x-v}{y-v} \right) q \leq (2k) \varepsilon^2 \leq (4k) \varepsilon^2.
\end{align*}
\]
- If \( z \lor q > x \) and \( z \lor q \leq x \) we reason as in the previous case.
- If \( z \lor q > x \) and \( z \lor q > y \), then 
\[
\begin{align*}
(\forall v \lor q)z & \leq (\forall v \lor q)(x \lor q) - (z \lor q, y) \\
& \leq \left( \frac{x-v}{y-v} \right) q \leq (4k) \varepsilon^2
\end{align*}
\]

Let \( \varepsilon^2 \) denote the equivalence class of \( \varepsilon \) modulo \( \theta \). Then \( A/\theta \models \varepsilon^2 = 0 \) but \( A/\theta \not\models \varepsilon = 0 \). Therefore \( A/\theta \) does not satisfy the axiom (A5) in Definition 3.1. It follows that the class of \( L_{\Pi} \)-algebras is not closed under quotients, hence it is not a variety.

**Lemma 3.3** Let \( A \) be any \( L_{\Pi} \)-algebra. Then for any \( x \in A \) and for any \( n, k \in N \setminus \{0\} \), if \( q^k x^n = 0 \) then \( x = 0 \).

**Proof.** Induction on \( k \). For \( k = 0 \) the claim follows from (A5). Suppose that the claim holds for \( k = m \), and let us prove it for \( k = m + 1 \). First note that letting \( x = 0 \) in axiom (A4) we get \( q \rightarrow q \rightarrow q = 0 \). Hence if \( qx = 0 \) then, by axiom (A4) one has \( x = q \rightarrow q qx = q \rightarrow q 0 = 0 \). So we have:

\[
qx = 0 \Rightarrow x = 0.
\]

Now if \( q^{m+1} x^n = 0 \) then \( 0 = q^{m+1} x^n = q(q^m x^n) \). Thus replacing \( x \) by \( q^m x^n \) in (1), we obtain \( q^m x^n = 0 \) and by the induction hypothesis, \( x = 0 \).

**Lemma 3.4**

(i) In any non-trivial \( L_{\Pi} \)-algebra one has \( q > 0 \).

(ii) Any linearly ordered \( L_{\Pi} \)-algebra has no zero divisors, i.e. if \( xy = 0 \), then either \( x = 0 \) or \( y = 0 \).

**Proof.** Claim (i) follows from Lemma 3.3, and claim (ii) follows from axiom (A5) of \( L_{\Pi} \)-algebras.
Theorem 3.5 Every subdirectly irreducible \( L \Pi_q \)-algebra is linearly ordered. Hence every \( L \Pi_q \)-algebra \( A \) can be decomposed as a subdirect product of a family of linearly ordered \( L \Pi_q \)-algebras.

Proof. For any \( a \in A \setminus \{0\} \), consider the family \( I_a \) of all MV-ideals \( J \) such that for every \( n, k \geq 0 \), \( q^k a^n \notin J \). \( I_a \) is non-empty, since by Lemma 3.3 \( \{0\} \in I_a \). Moreover \( I_a \) is closed under unions of chains, therefore \( (I_a, \subseteq) \) is an inductive partially ordered set, and, by Zorn’s lemma, it has a maximal element, call it \( J_a \).

Let \( A^- \) be the PMV-reduct of \( A \). Since \( A \) is a PMV-algebra and its MV-reduct have the same congruences, the congruence \( \theta_a \), associated with \( J_a \) is a congruence of PMV-algebras, too. Therefore \( A^-/\theta_a \) is a PMV-algebra. To continue the proof we show the following lemmas.

Lemma 3.6 For every \( b, c \in A \), either \( b \cap c \in J_a \) or \( c \cap b \in J_a \).

Proof. Let by contradiction \( b, c \in A \) be such that \( b \cap c \notin J_a \) and \( c \cap b \notin J_a \). Let for any subset \( X \) of \( A^- \), \( X \) denote the ideal generated by \( X \). By the maximality of \( J_a \) there exist \( k, n, h, m > 0 \) with \( q^k a^n \in J_a \cup \{b \cap c\} \) and \( q^h a^m \in J_a \cup \{c \cap b\} \). Thus there are \( f, g \in J_a \) and \( r, s \in N \) such that

\[
q^k a^n \leq f \oplus (r)(b \cap c) \quad \text{and} \quad q^h a^m \leq g \oplus (s)(c \cap b).
\]

Let \( u = f \lor g \) and \( t = \max\{k, n, h, m, r, s\} \). Then

\[
q^t a^t \leq u \lor (r)(b \cap c) \quad \text{and} \quad q^t a^t \leq u \lor (s)(c \cap b),
\]

therefore \( q^t a^t \leq u \lor ((r)(b \cap c) \land (s)(c \cap b)) = u \) and \( q^t a^t \in J_a \), which is a contradiction. \( \blacksquare \)

Lemma 3.7 If \( bc \in J_a \) then either \( b \in J_a \) or \( c \in J_a \).

Proof. Let by contradiction \( b, c \in A \) be such that \( b \notin J_a \), \( c \notin J_a \) and \( bc \in J_a \). By the maximality of \( J_a \) there exist \( h, k, m, n > 0 \) such that

\[
q^k a^n \in J_a \cup \{b\} \quad \text{and} \quad q^h a^m \in J_a \cup \{c\}.
\]

Thus there are \( f, g \in J_a \) and \( r, s \in N \) such that \( q^k a^n \leq f \oplus (r)b \) and \( q^h a^m \leq g \oplus (s)c \).

Let \( u = f \lor g \) and \( t = \max\{h, k, m, n, r, s\} \). Then \( q^t a^t \leq u \lor (t)b \) and \( q^t a^t \leq u \lor (t)c \), therefore \( q^{2t}a^{2t} \leq (u \lor (t)b)(u \lor (t)c) = u^2 \lor ((t)uc) \lor ((t)ub) \lor ((t^2)bc) \). Now \( u^2 \lor ((t)uc) \lor ((t)ub) \in J_a \), and \( (t^2)bc \in J_a \), therefore \( q^{2t}a^{2t} \in J_a \), and a contradiction has been reached. \( \blacksquare \)

We continue the proof of theorem 3.5. Since \( a \notin J_a, \cap_{a \in A \setminus \{0\}} J_a = \{0\} \), hence \( \bigcap_{a \in A \setminus \{0\}} \theta_a \) is the minimal congruence. It follows that the map

\[
\Phi : A^- \overset{\Phi}{\longrightarrow} \prod_{a \in A \setminus \{0\}} A^-/\theta_a \quad \text{defined by} \quad \Phi(b) = \langle b/\theta_a : a \in A \setminus \{0\} \rangle
\]
is a monomorphism from $A^{-}$ to $\prod_{a \in A \setminus \{0\}} A/\theta_a$.
In other words $A^{-}$ can be decomposed as a subdirect product of linearly ordered PMV-algebras. Moreover by Lemma 3.7, each component $A/\theta_a$ has no zero divisors. Finally, $q/\theta_a \neq 0$, because $q \not\in J_a$. Thus we have shown:

**Lemma 3.8** The PMV-reduct of any LII$_q$-algebra can be decomposed as a subdirect product of a family of linearly ordered PMV-algebras $\langle A_i : i \in I \rangle$ without zero divisors. Moreover, for every $i \in I$, $q_i > 0$. ■

**Lemma 3.9** For any $a, b \in A$ and for every $i \in I$, the following conditions hold:

If $a_i \vee q_i \leq b_i$, then $(a \rightarrow_q b)_i = 1$.

Otherwise, $(a \rightarrow_q b)_i$ is the unique $z_i \in A_i$ such that $(a_i \vee q_i)z_i = b_i$.

In particular $(a \rightarrow_q b)_i$ depends on $a_i$ and $b_i$ but not on $a$ and $b$.

**Proof.** First of all recall that $(a \vee q)(a \rightarrow_q b) = (a \vee q)(a \rightarrow_q b) = b \wedge (a \vee q)$. Hence for every $i \in I$ we have $(a_i \vee q_i)(a \rightarrow_q b)_i = b_i \wedge (a_i \vee q_i)$. Let $z_i = (a \rightarrow_q b)_i$. Then:

If $(a \vee q)_i \leq b_i$ then $(a \vee q)_i z_i = ((a \vee q) \wedge b)_i = (a \vee q)_i$. So $(a \vee q)_i \wedge (a \vee q)_i z_i = (a \vee q)_i (1 \wedge z_i = 0$. Since $(a \vee q)_i > 0$ and $A_i$ has no zero divisors (Lemma 3.8) we get $z_i = 1$.

If $(a \vee q)_i > b_i$, then $(a \vee q)_i z_i = b_i$. Moreover, $z_i$ is the unique element with this property. Indeed if $\langle a \vee q \rangle u = b_i$ then $(a \vee q)_i | u - z_i | = 0$, and since $A_i$ has no zero divisors and $q_i > 0$ we conclude that $u = z_i$.

We conclude the proof of Theorem 3.5. Define for $a, b \in A$ and for $i \in I$, $a_i \rightarrow_q b_i = (a \rightarrow_q b)_i$. By Lemma 3.9 this definition is admissible. By Lemma 3.8 and 3.9, $A_i$ equipped by the additional operator $\rightarrow_q$ satisfies axioms (A1) $\ldots$ (A3) and (A5) of LII$_q$-algebras. Let us check axiom (A4). If $x = 1$ then $q_i \rightarrow_i q_i x = 1 = x$. Otherwise, $q_i x < q_i$ and by Lemma 3.9, $q_i \rightarrow_i q_i x$ is the unique $z$ such that $q_i z = x$. But $q_i z = q_i x$ implies $z = x$, therefore $q_i \rightarrow_i q_i x = x$. This concludes the proof.

**Corollary 3.10** Every LII$_q$-algebra is a subalgebra of a $q$-reduct of a LII-algebra.

**Proof.** Let $A$ be any LII$_q$-algebra, and let $A_i : i \in I$ be the linearly ordered factors in the subdirect representation of $A$ according to Theorem 3.5, let $A_i^{-}$ denote the PMV-reduct of $A_i$, and let $\Gamma_R$ be the functor defined in Section 2. Then by Proposition 2.6 for every $i \in I$ there is a c-s-u-f-ring $R_i$ such that $A_i^{-} = \Gamma_R(R_i)$. It is readily seen that $R_i$ is linearly ordered (because $\Gamma_R(R_i)$ is linearly ordered). Moreover $R_i$ has no zero divisors: if $x \times y = 0$, then letting $|x| = x \vee -x$, and $z = \min\{1, |x|, |y|\}$ we have $z \in \Gamma_R(R_i) = A_i$, and $z^2 = 0$. 8
By axiom (A5) this implies z = 0. This is only possible if either x = 0 or y = 0. It follows that the ring reduct of $R_i$ is an integral domain. Now let $F_i$ be the fraction field of $R_i$. Then $A_i$ is a subalgebra of $\Gamma_R(F_i)$. For $x, y \in \Gamma_R(F_i)$, define

\[(x \rightarrow_{\pi} y)_i = \begin{cases} 1 & \text{if } x \leq y \\ yx^{-1} & \text{otherwise} \end{cases}\]

Then $\rightarrow_{\pi}$ makes $\Gamma_R(F_i)$ a LIΠ-algebra (see [Mo00]), call it $LP_i$. Moreover by Lemma 3.9 for all $x, y \in A$ we have: $x \rightarrow_q y = (x \lor q) \rightarrow_{\pi} y$. Therefore $A_i$ is a subalgebra of a q-reduct of $LP_i$, and $A$ is a subalgebra of a q-reduct of $\prod_{i \in I} LP_i$.

**Definition 3.11** Let $A$ be any LIΠ$_q$-algebra. We say that $\varepsilon \in A \setminus \{0\}$ is an infinitesimal if for any natural number $n$ one has: $(n)\varepsilon \leq -\varepsilon$.

The next corollary shows that any linearly ordered LIΠ$_q$-algebra which is not a q-reduct of a LIΠ-algebra must have infinitesimals:

**Corollary 3.12** Let A be a linearly ordered LIΠ$_q$-algebra without infinitesimals. Then $A$ is a q-reduct of an LIΠ-algebra.

**Proof.** We just need to check that product in $A$ has a residual $\rightarrow_{\pi}$. This amounts to prove that for any $x, y$ there is a $z$ such that $zx = x \land y$. If $x \leq y$ then we can take $z = 1$. If $x = 1$, then we can take $z = y$. Otherwise, since there are no infinitesimals, there is $n \in N$ such that $(n)x \geq -x$. Take $n$ minimal with this property. Now recall that $A$ embeds into a q-reduct of a linearly ordered LIΠ-algebra $B$ (Corollary 3.10), and that every linearly ordered LIΠ-algebra embeds into an ultrapower of the LIΠ-algebra $[0, 1]_{LI}$ on $[0, 1]$ ([Mo02]). Hence the universal formula

$$\forall x \forall y((\forall x (x \geq -x) \& \forall y ((n-1)x < -x) \& \forall y < x)) \Rightarrow (x \rightarrow_{\pi} y = (n)x \rightarrow_{\pi} (n)y)$$

(where $\&$ and $\Rightarrow$ denote classical conjunction and classical implication respectively) being true in $[0, 1]_{LI}$, is true in $B$. Now $q \leq (n)x$ (because $q \leq -q$). Since $A$ embeds into a q-reduct of $B$, $(n)x \rightarrow_q (n)y = ((n)x \lor q) \rightarrow_{\pi} (n)y = (n)x \rightarrow_{\pi} (n)y = x \rightarrow_{\pi} y$.

### 4 Generation by standard LIΠ$_q$-algebras

This section is entirely devoted to the proof of the fact that the variety of LIΠ$_q$-algebras is generated as a quasivariety by its standard members, i.e., by those LIΠ$_q$-algebras whose lattice reduct is $([0, 1], \max, \min)$.

**Definition 4.1** In the sequel, for every $0 < q \leq \frac{1}{2}, [0, 1]_q$ will denote the LIΠ$_q$-algebra $([0, 1], \lor, \land, \neg, q, 0, 1, q)$, where $\lor$, $\land$ and $\neg$ are defined as usual, and $x \rightarrow_q y = (x \lor q) \rightarrow_{\pi} y = \begin{cases} \frac{x}{x \lor q} & \text{if } x \lor q > y \\ 1 & \text{otherwise} \end{cases}$
Theorem 4.2 The class of $LII_q$-algebras is generated as a quasivariety by the class $S = \{[0, 1)_q : 0 < q \leq \frac{1}{2}\}$.

Proof. Let $\Phi$ be a quasi identity which is not valid in all $LII_q$-algebras. Then $\Phi$ fails to hold in some subdirectly irreducible, hence (Theorem 3.5) linearly ordered, $LII_q$-algebra $A$. Now (Corollary 3.10) $A$ embeds into a $q$-reduct $B$ of a linearly ordered $LII$-algebra $D$, and $\Phi$ fails in $B$, too. Moreover, ([Mo01]) every linearly ordered $LII$-algebra embeds into an ultrapower $E$ of the $LII$-algebra $[0, 1]_{LII}$ on $[0, 1]$. At this point, we can observe that the existence of an evaluation $e$ in $B$ which invalidates $\Phi$ can be written as an existential formula (in the language of $LII$-algebras) of the form

$$\exists q \exists x_1 \ldots \exists x_n (0 < q \& q \leq \neg q \& \Psi(x_1, \ldots, x_n, q))$$

where $\Psi$ quantifier-free, and $x_1, \ldots, x_n$ are the variables occurring in $\Phi$. Such a formula is preserved under taking superstructures, hence it is true in $E$, and finally it is true in $[0, 1]_{LII}$. Let $q \in (0, \frac{1}{2}]$ and $a_1, \ldots, a_n \in [0, 1]$ be such that $\Psi(a_1, \ldots, a_n, q)$ is true in $[0, 1]_{LII}$, and let $e$ be the evaluation defined by $e(a_i) = a_i$ for $i = 1, \ldots, n$. Then $\Phi$ is invalidated by $e$ in $[0, 1]_q$.

Corollary 4.3 Let $A$ be a linearly ordered $LII_q$-algebra with more than two elements. Then the $PMV$-reduct $A^-$ of $A$ has a subalgebra isomorphic to $(\mathbb{Q} \cap [0, 1], \oplus, \neg, \cdot, 0, 1)$.

Proof. By Corollary 3.10, $B$ is a subalgebra of a $q$-reduct of a linearly ordered $LII$-algebra $D$. Hence it is sufficient to prove that for all $n \in \mathbb{N} \setminus \{0\}$ there is an element $a$ of $B$, denoted by $\frac{1}{n}$, such that $(n - 1)a = \neg a$. Indeed if we prove this, then as in [Mo00] we can see that the map $\Phi : \frac{m}{n} \mapsto (m)\frac{1}{n}$ is an embedding of $(\mathbb{Q} \cap [0, 1], \oplus, \neg, \cdot, 0, 1)$ into the $PMV$-reduct of $B$. Let $h = (q \oplus q) \rightarrow q$. Then $h = \neg h$, because this property can be expressed by an equation which is true in any $q$-reduct of $[0, 1]_{LII}$, hence by Theorem 4.2 it is true in any $LII_q$-algebra. Let $k$ be the minimum natural number such that $2^k \geq n$, and let $a = (n)h^k \rightarrow q h^k$. Then for any choice of $0 < q \leq h$ we have that $q \leq h \leq (n)h^k$. Hence $a = (n)h^k \rightarrow q h^k = (n)h^k \rightarrow (n)h^k$, and $(n - 1)a = \neg a$. Since this fact can be expressed by a universal Horn formula, it holds in any $LII$-algebra. Hence $(n - 1)a = \neg a$, and we can take $\frac{1}{n} = a$.

References


