Representation of Partial Traces

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Traces in symmetric monoidal categories

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![Diagram](image-url)
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**Partial traces**

**P. Scott & E. Haghverdi:** axiomatization of partially-defined trace, capturing the idea of (partially defined) categorical feedback.

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A straightforward way to build partial traces:
Partial traces and sub-categories

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○ Consider a totally traced category $\mathcal{D}$.

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**Definition**

Define a partial trace $\hat{\text{Tr}}_U$ on $\mathcal{C}$ as:

- If $\text{Tr}_U[f] \in \mathcal{C}$, then $\hat{\text{Tr}}_U[f] = \text{Tr}_U[f]$, undefined otherwise.
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\[ C \xrightarrow{E_C} T(C) \]

(where \( C \) is partially traced, \( T(C) \) is the totally traced category in which it embeds, \( D \) is any other totally traced category, with \( F \) a traced functor from \( C \) to \( D \) )
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D & \xrightarrow{G} & D
\end{array}
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Original proof: intermediate partial version of the \( \text{Int}(\cdot) \) construction and "paracategories".

Contribution: a more direct and simplified proof.
The proof (I): the dialect construction

A generic construction $D(C)$ on any monoidal category $C$.
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- $U$ an object of $C$.
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When composing $(f, U)$ and $(g, V)$ the state spaces do not interact.
The proof (II): hiding and congruences

**Hiding:** given a partially traced $C$ we can look at $D(C)$ and define a *hiding* operation
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$$H_U[f, V] = (f, U \otimes V) : A \to B$$

$H$ behaves a lot like a (total) trace.

Congruences: consider the equivalence relation on morphisms generated by some required equations, including $$(f, U \otimes V) \approx (\text{Tr}_V[f], U)$$ when $\text{Tr}_V[f]$ is defined. Then we can set $T(C) = D(C)/\approx$ in which $H$ induces a total trace, encompassing the original partial trace of $C$. 
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The proof (II): hiding and congruences

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Then we can set $T(C) = D(C) / \approx$ in which $H[\cdot]$ induces a total trace, encompassing the original partial trace of $C$. 
We can embed $C$ in $T(C)$ by setting $E_C(f) = (f, 1)$. 

Is it really an embedding? We check that $(f, 1) \approx (g, 1)$ implies $f = g$. Because $\approx$ is freely generated, we can do it by induction on chains of elementary equivalences.

Universal property: we can close the diagram

\[
\begin{array}{c}
\text{C} \\
\downarrow \\
\text{T(C)} \\
\downarrow \\
\text{D} \\
\downarrow \\
\text{E} \\
\downarrow \\
\text{C} \\
\end{array}
\]

by setting $G(f, U) = 
\text{Tr}_F U \left[ F f \right]$. 

(well defined because $(f, U) \approx (g, V)$ implies $\text{Tr}_F U \left[ F f \right] = \text{Tr}_F V \left[ F g \right]$).

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\Big\downarrow & & \Big\downarrow \\
\mathcal{D} & \overset{F}{\longrightarrow} & \\
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Conclusion

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Allows intuitive diagrammatic reasoning also in the partially-defined case.
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... Thank you for your attention